7 Planar systems of linear ODE

Here I restrict my attention to a very special class of autonomous ODE: linear ODE with constant coefficients. This is arguably the only class of ODE for which explicit solution can always be constructed. Linear systems considered as mathematical models of biological processes are of limited use; however, such models still can be used to describe the dynamics of the system during the stages when the interactions between the elements of the system can be disregarded. Moreover, the analysis of the linear systems is a necessary step in analysis of a local behavior of nonlinear systems (linearization of the system in a neighborhood of an equilibrium).

7.1 General theory

The linear system of first order ODE with constant coefficients on the plane has the form

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2,\\ \dot{x}_2 &= a_{21}x_2 + a_{22}x_2,\end{aligned} \tag{1}$$

or, in the vector notations,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^2, \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2)^\top$, and $\boldsymbol{A} = (a_{ij})_{2\times 2}$ is a matrix with real entries:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Additionally to (2), consider also the initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{3}$$

To present the solution to (2)–(3), I first prove

Proposition 1. *Initial value problem* (2)–(3) *is equivalent to the solution of the integral equation*

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}(\xi)\,\mathrm{d}\xi. \tag{4}$$

Proof. Assume that $\boldsymbol{x}$ solves (4). Then, by direct inspection I have that $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Moreover, since the right-hand side is given by the integral this implies that $\boldsymbol{x} \in \mathcal{C}^{(1)}$, therefore, I can take the derivative to find (2). Now other way around, by assuming that $\boldsymbol{x}$ solves (2)–(3), integrating (2), and evaluating the constant of integration, I recover (4). ∎

Now I can use (4) to approximate the solution to (2)–(3) by the *method of successive iterations.* The first approximation is of course the initial condition $\boldsymbol{x}_0$:

$$\boldsymbol{x}_1(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}_0\,\mathrm{d}\xi = \boldsymbol{x}_0 + \boldsymbol{A}\boldsymbol{x}_0 t.$$

Math 484/684: Mathematical modeling of biological processes by Artem Novozhilov

e-mail: artem.novozhilov@ndsu.edu. Fall 2015

Next,

$$\boldsymbol{x}_2(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}_1(\xi)\,\mathrm{d}\xi = \boldsymbol{x}_0 + \boldsymbol{A}\boldsymbol{x_0}t + \frac{\boldsymbol{A}^2\boldsymbol{x}_0 t^2}{2}\,.$$

Or, continuing the process,

$$\boldsymbol{x}_n(t) = \left(\boldsymbol{I} + \frac{\boldsymbol{A}t}{1!} + \frac{\boldsymbol{A}^2t^2}{2!} + \ldots + \frac{\boldsymbol{A}^n t^n}{n!}\right)\boldsymbol{x}_0.$$

Here $\boldsymbol{I}$ is the identity matrix. Please note that what is inside the parenthesis in the last formula is actually a matrix. However, the expression is so suggestive, given that you remember the Taylor series for the ordinary exponential function,

$$\exp(t) = e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots,$$

that it is impossible to resist the temptation to make

Definition 2. *The matrix exponent of the matrix $\boldsymbol{A}$ is defined as the infinite series*

$$\exp(\boldsymbol{A}) = e^{\boldsymbol{A}} = \boldsymbol{I} + \frac{\boldsymbol{A}}{1!} + \frac{\boldsymbol{A}^2}{2!} + \ldots + \frac{\boldsymbol{A}^n}{n!} + \ldots \tag{5}$$

This definition suggests that the solution to the integral equation (4), and, therefore, to the IVP (2)–(3), is of the form

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0,$$

note that here I have to write $\boldsymbol{x}_0$ on the right to make sure all the operations are well defined. Before continuing the analysis of the linear system and actually proving that indeed the solution is given by the presented formula, I have to make sure that the given definition makes sense, i.e., all the usual series (there are four of them if matrix $\boldsymbol{A}$ is 2×2) converge.

Proposition 3. *Series* (5) *converges absolutely.*

Proof. Let $|a_{ij}| \leq a$. For the product $\boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}^2$ I have that the elements of this product are bonded by $2a^2$. Similarly, for $\boldsymbol{A}^k$ it is $2^{k-1}a^k = 2^k a^k/2$. Since I have that

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k a^k}{k!} = \frac{1}{2}e^{2a},$$

therefore the series in (5) converges absolutely to the matrix denoted $e^{\boldsymbol{A}}$. ■

Now it is quite straightforward to prove that $\exp(\boldsymbol{A}t)$ solves (2).

Proposition 4.

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}.$$

Proof. Since the series converges absolutely, I am allowed to differentiate the series termwise:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{A}^k t^k}{k!} = \sum_{k=1}^{\infty}\frac{\boldsymbol{A}^k t^{k-1}}{(k-1)!} = \boldsymbol{A}e^{\boldsymbol{A}t}.$$

■

The last proposition indicates that the matrix exponent has some properties similar to the usual exponent. Here is a good example to be careful when dealing with the matrix exponent.

Example 5. Consider two matrices,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

I claim that

$$e^{\boldsymbol{A}+\boldsymbol{B}} \neq e^{\boldsymbol{A}} e^{\boldsymbol{B}}.$$

Let me prove it. I have

$$\boldsymbol{A}^2 = 0,$$

therefore

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly,

$$e^{\boldsymbol{B}} = \boldsymbol{I} + \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$e^{\boldsymbol{A}} e^{\boldsymbol{B}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Now

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

I have

$$\boldsymbol{C}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\boldsymbol{I}, \quad \boldsymbol{C}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\boldsymbol{C}, \quad \boldsymbol{C}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}.$$

Therefore,

$$e^{\boldsymbol{C}} = \begin{bmatrix} 1 - \frac{1}{2!} + \frac{1}{4!} + \ldots & 1 - \frac{1}{3!} + \frac{1}{5!} + \ldots \\ -1 + \frac{1}{2!} - \frac{1}{4!} + \ldots & 1 - \frac{1}{2!} + \frac{1}{4!} + \ldots \end{bmatrix} = \begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix},$$

which proves that $e^{\boldsymbol{A}+\boldsymbol{B}} \neq e^{\boldsymbol{A}} e^{\boldsymbol{B}}$.

In the last expression I used

$$\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}, \quad \sin t = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!}.$$

Proposition 6. *If*

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{A} = 0,$$

i.e., if matrices $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *commute, then*

$$e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t} e^{\boldsymbol{B}t}.$$

Proof. For the matrices that commute the binomial theorem holds:

$$(\boldsymbol{A}+\boldsymbol{B})^n = \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{A}^{n-k}\boldsymbol{B}^k = n! \sum_{i+j=n} \frac{\boldsymbol{A}^i}{i!}\frac{\boldsymbol{B}^j}{j!}\,.$$

Since, by the Cauchy product

$$\left(\sum_{i=0}^{\infty}\frac{\boldsymbol{A}^i}{i!}\right)\left(\sum_{j=0}^{\infty}\frac{\boldsymbol{B}^j}{j!}\right) = \sum_{n=0}^{\infty}\sum_{i+j=n}\frac{\boldsymbol{A}^i}{i!}\frac{\boldsymbol{B}^j}{j!}\,,$$

one has

$$e^{\boldsymbol{A}}e^{\boldsymbol{B}} = \sum_{n=0}^{\infty}\sum_{i+j=n}\frac{\boldsymbol{A}^i}{i!}\frac{\boldsymbol{B}^j}{j!} = \sum_{n=0}^{\infty}\frac{(\boldsymbol{A}+\boldsymbol{B})^n}{n!} = e^{\boldsymbol{A}+\boldsymbol{B}}.$$

I can do these formal manipulations since all the series in the question converge absolutely. ■

As an important corollary of the last proposition I have

Corollary 7. *For the matrix exponent*

$$e^{\boldsymbol{A}(t_1+t_2)} = e^{\boldsymbol{A}t_1}e^{\boldsymbol{A}t_2},$$

and

$$e^{\boldsymbol{A}t}e^{-\boldsymbol{A}t} = \boldsymbol{I}.$$

Proof. For the first note that $\boldsymbol{A}t_1$ and $\boldsymbol{A}t_2$ commute. For the second put $t_1 = t$ and $t_2 = -t$. ■

Now I can actually prove that the solution to (2)–(3) exists, unique, and defined for all $-\infty < t < \infty$.

Theorem 8. *Solution to the IVP problem* (2)–(3) *is unique and given by*

$$\boldsymbol{x}(t;\boldsymbol{x}_0) = e^{\boldsymbol{A}t}\boldsymbol{x}_0, \quad -\infty < t < \infty. \tag{6}$$

Proof. First, due to Proposition 4,

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t}\boldsymbol{x}_0 = \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}_0,$$

and also $e^{\boldsymbol{A}0}\boldsymbol{x}_0 = \boldsymbol{I}\boldsymbol{x}_0 = \boldsymbol{x}_0$, which proves that (6) is a solution. To prove that it is unique, consider any solution $\boldsymbol{x}$ of the IVP and put

$$\boldsymbol{y}(t) = e^{-\boldsymbol{A}t}\boldsymbol{x}(t).$$

I have

$$\begin{aligned}\dot{\boldsymbol{y}}(t) &= -\boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t)\\ &= -\boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x} = 0.\end{aligned}$$

In the expression above, I used two (non-obvious) facts: First, that the usual product rule is still true for the derivative of the product of two matrices, and that

$$e^{-\boldsymbol{A}t}\boldsymbol{A} = \boldsymbol{A}e^{-\boldsymbol{A}t},$$

i.e., that matrices $\boldsymbol{A}$ and $e^{-\boldsymbol{A}t}$ commute. You should fill in the details of the proofs of these statements.

Hence I have that $\boldsymbol{y}(t) = C$, therefore, by setting $t = 0$, I find $\boldsymbol{y}(0) = \boldsymbol{x}_0$, which implies that any solution is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

Since the last formulae is defined for any t, therefore, the solution is defined for $-\infty < t < \infty$. ∎

7.2 Three main matrices and their phase portraits

Consider the solutions to (2) and their phase portraits for three matrices:

$$\boldsymbol{A}_1 = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix},$$

where $\lambda_1, \lambda_2, \lambda, \alpha, \beta$ are real numbers.

- $\boldsymbol{A}_1$. I have, using the definition of the matrix exponent, that

  $$e^{\boldsymbol{A}_1 t} = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix},$$

  therefore the general solution to (2) is given (which can be actually obtained directly, by noting that the equations in the system are *decoupled*)

  $$\boldsymbol{x}(t;\boldsymbol{x}_0) = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\boldsymbol{x}_0 = \begin{bmatrix}e^{\lambda_1 t}x_1^0\\ e^{\lambda_2 t}x_2^0\end{bmatrix}.$$

  If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ then I have only one isolated equilibrium $\hat{\boldsymbol{x}} = (0,0)$, the phase curves can be found as solutions to the first order ODE

  $$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\lambda_2 x_2}{\lambda_1 x_1},$$

  which is separable equation, and the directions on the orbits are easily determined by the signs of λ_1 and λ_2 (i.e., if $\lambda_1 < 0$ then $x_1(t) \to 0$ as $t \to \infty$).

  Consider a specific example with $0 < \lambda_1 < \lambda_2$. In this case I have that all the orbits are parabolas, and the direction is from the origin because both lambdas are positive. The only tricky part here is to determine which axis the orbits approach as $t \to -\infty$, this can be done by looking at the explicit equations for the orbits (you should do it) or by noting that when $t \to -\infty$ $e^{\lambda_1 t} \gg e^{\lambda_2 t}$ and therefore x_1 component dominates in a small enough neighborhood of $(0,0)$ (see the figure). The obtained phase portrait is called *topological node* ("topological" is often dropped), and since the arrows point from the origin, it is unstable (I will come back to the discussion of the stability a little later).

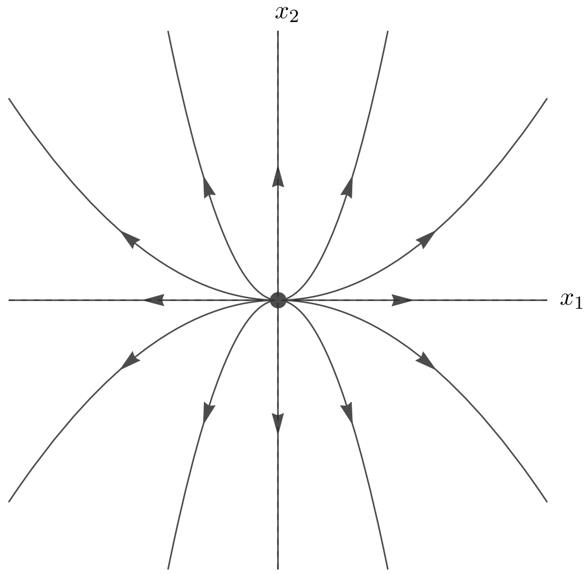

Figure 1: Unstable node in system (2) with matrix $\boldsymbol{A}_1$. The eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ coincide with the directions of x_1-axis and x_2-axis respectively

As another example consider the case when $\lambda_2 < 0 < \lambda_1$. In this case (prove it) the orbits are actually hyperbolas on (x_1, x_2) plane, and the directions on them can be identifies by noting that on x_1-axis the movement is *from* the origin, and on x_2-axis it is *to* the origin. Such phase portrait is called *saddle*. All the orbits leave a neighborhood of the origin for both $t \to \pm\infty$ except for five special orbits: first, this is of course the origin itself, second two orbits on x_1-axis that actually approach the origin if $t \to -\infty$, and two orbits on x_2-axis, which approach the origin if $t \to \infty$. The orbits on x_1-axis form the *unstable manifold* of $\hat{\boldsymbol{x}} = (0, 0)$, and the orbits on x_2-axis form the *stable manifold* of $\hat{\boldsymbol{x}}$. These orbits are also called the saddle's *separatrices* (singular, *separatrix*).

There are several other cases, which need to be analyzed, let me list them all:

- $0 < \lambda_1 < \lambda_2$: unstable node (shown in figure)
- $0 < \lambda_2 < \lambda_1$: unstable node
- $0 < \lambda_1 = \lambda_2$: unstable node
- $\lambda_1 < \lambda_2 < 0$: stable node
- $\lambda_2 < \lambda_1 < 0$: stable node
- $\lambda_1 = \lambda_2 < 0$: stable node
- $\lambda_1 < 0 < \lambda_2$: saddle
- $\lambda_2 < 0 < \lambda_1$: saddle (shown in figure)

You should sketch the phase portraits for each of these cases. Also keep in mind that for now I exclude cases when one or both λ's are zero.

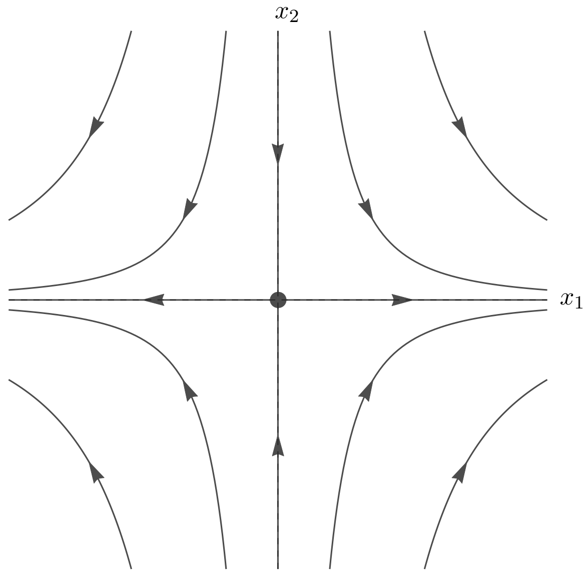

Figure 2: Saddle in system (2) with matrix $\boldsymbol{A}_1$. Eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ coincide with the directions of x_1-axis and x_2-axis respectively

- $\boldsymbol{A}_2$. To find $e^{\boldsymbol{A}_2 t}$ I will use the fact that

$$\boldsymbol{A}_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  and these two matrices commute. The matrix exponent for the first matrix was found in the previous point, and for the second it is readily seen that

$$\exp \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

  Therefore,

$$e^{\boldsymbol{A}_2 t} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

  Assume that $\lambda < 0$ (the cases $\lambda = 0$ and $\lambda > 0$) left as exercises. Now, first, we see that $\boldsymbol{x}(t; \boldsymbol{x}_0) \to 0$ as $t \to \infty$, moreover,

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} \to 0$$

  as $t \to \infty$, therefore the orbits should be tangent to x_1-axis. The figure is given below, the phase portrait is sometimes called the *improper stable node.*

- $\boldsymbol{A}_3$. Here I will use

$$\boldsymbol{A}_3 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

  as two commuting matrices to find

$$e^{\boldsymbol{A}_3 t} = e^{\alpha t} \begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix}.$$

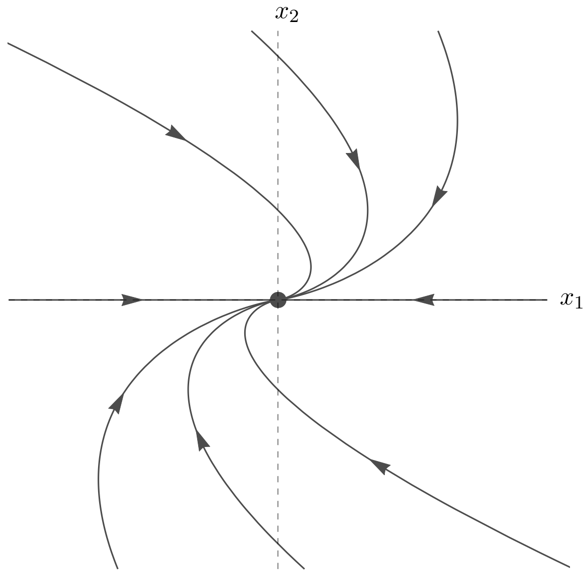

Figure 3: Improper node in system (2) with matrix $\boldsymbol{A}_2$. Vector $\boldsymbol{v}_1$ coincides with the direction of x_1-axis

Hence the flow of the system (2) is given by

$$\boldsymbol{x}(t;\boldsymbol{x}_0) = e^{\boldsymbol{A}_3 t}\boldsymbol{x}_0 = e^{\alpha t}\begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}\boldsymbol{x}_0.$$

To determine the phase portrait observe that if $\alpha < 0$ then all the solutions will approach the origin, and if $\alpha > 0$, they will go away from origin. We also have components of $e^{\boldsymbol{A}_3 t}$ which are periodic functions of t, which finally gives us the whole picture: if $\alpha < 0$ and $\beta > 0$ then the orbits are the spirals approaching origin clockwise, if $\alpha > 0$ and $\beta > 0$ then the orbits are spiral unwinding from the origin clockwise, and if $\alpha = 0$ then the orbits are closed curves. Here is an example for $\alpha < 0$ and $\beta < 0$, this phase portrait is called the *stable focus* (or *spiral*).

If I take $\alpha = 0$ and $\beta < 0$ then the phase portrait is composed of the closed curves and called the center (recall the Volterra–Lotka model). See the figure.

To determine the direction on the orbits, I can use the original vector field. For example, in the case $\alpha = 0$ $\beta < 0$ I have that for any point $x_1 = 0$ and $x_2 > 0$ the derivative of x_2 is negative, and therefore the direction is counter-clockwise.

7.3 A little bit of linear algebra

So why actually did I spend so much time on studying three quite simple particular matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$? It is because the following theorem is true:

Theorem 9. *Let $\boldsymbol{A}$ be a 2×2 real matrix. Then there exists a real invertible 2×2 matrix $\boldsymbol{P}$ such that*

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{J},$$

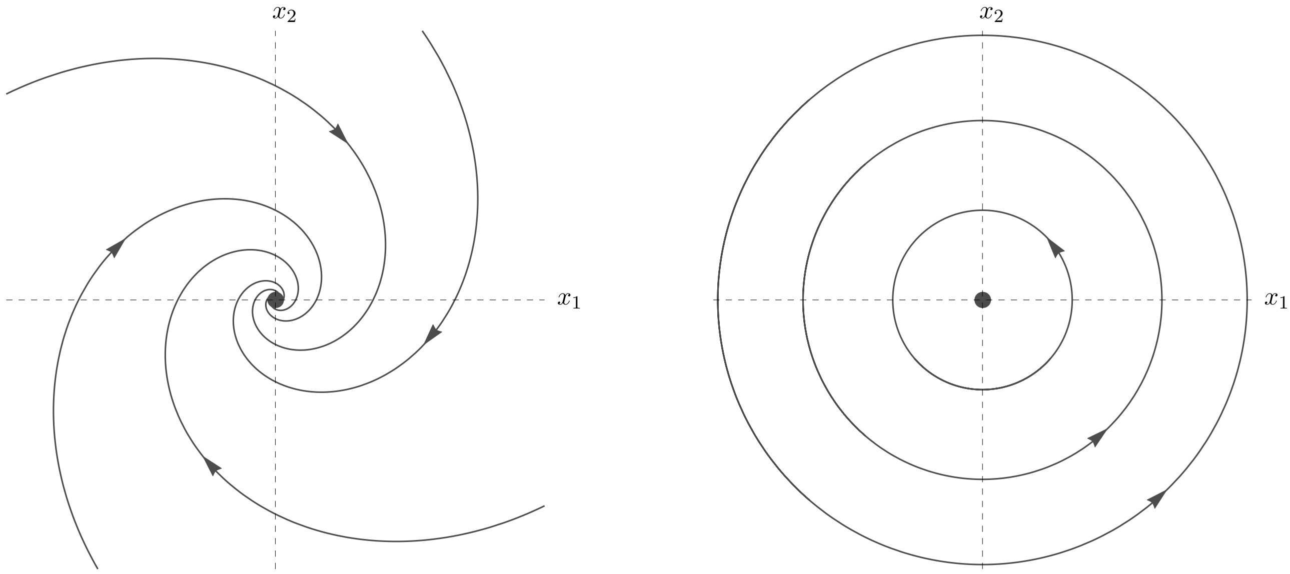

Figure 4: Left: Stable focus (spiral) and right: center in system (2) with matrix $\boldsymbol{A}_3$

where matrix $\boldsymbol{J}$ is one of the following three matrices in Jordan's normal form

$$(a)\quad \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix},\quad (b)\quad \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix},\quad (c)\quad \begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}.$$

Before turning to the proof of this theorem, let me discuss how this theorem can be used for the analysis of a general system (2):

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^2. \tag{2}$$

Consider a linear change of variables $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$ for the new unknown vector $\boldsymbol{y}$. I have

$$\boldsymbol{P}\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{y},$$

or

$$\dot{\boldsymbol{y}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{y} = \boldsymbol{J}\boldsymbol{y},$$

therefore,

$$\boldsymbol{y}(t;\boldsymbol{y}_0) = e^{\boldsymbol{J}t}\boldsymbol{y}_0,$$

where I already know how to calculate $e^{\boldsymbol{J}t}$. Returning to the original variable, I find that

$$\boldsymbol{x}(t;\boldsymbol{x}_0) = \boldsymbol{P}e^{\boldsymbol{J}t}\boldsymbol{P}^{-1}\boldsymbol{x}_0,$$

full solution to the original problem. Moreover, I also showed that

$$e^{\boldsymbol{A}t} = \boldsymbol{P}e^{\boldsymbol{J}t}\boldsymbol{P}^{-1},$$

which is often used to calculate the matrix exponent. Finally, the phase portraits for $\boldsymbol{x}$ will be similar to those of $\boldsymbol{y}$, since the linear invertible transformation amounts to scaling, rotation, and reflection, as we are taught in the course of linear algebra. The only question is how to find this linear change $\boldsymbol{P}$.

For this, let me recall

Definition 10. *A nonzero vector $\boldsymbol{v}$ is called an eigenvector of matrix $\boldsymbol{A}$ if*

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v},$$

where λ is called the corresponding eigenvalue.

Generally eigenvalues and eigenvectors can be complex. To find the eigenvalues, I need to find the roots of the characteristic polynomial

$$P(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}),$$

which is of the second degree if $\boldsymbol{A}$ is a 2×2 matrix. Once the eigenvalues are found, the corresponding eigenvectors can be found as solutions to the homogeneous system of linear algebraic equations

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0.$$

Remember that eigenvectors are not unique and determined up to a multiplicative constant. Now I am in a position to prove Theorem 9. The proof also gives the way to find the transformation $\boldsymbol{P}$.

Proof of Theorem 9. Since the characteristic polynomial has degree two, it may have either two real roots, two complex conjugate roots, or one real root multiplicity two.

I assume first that I either have two distinct real roots $\lambda_1 \in \mathbf{R} \neq \lambda_2 \in \mathbf{R}$ with the corresponding eigenvectors $\boldsymbol{v}_1 \in \mathbf{R}^2$ and $\boldsymbol{v}_2 \in \mathbf{R}^2$ or a real root $\lambda \in R$ multiplicity two which has two linearly independent eigenvectors $\boldsymbol{v}_1 \in \mathbf{R}^2$ and $\boldsymbol{v}_2 \in \mathbf{R}^2$. Now I consider the matrix $\boldsymbol{P}$, whose columns are exactly $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, I will use the notation

$$\boldsymbol{P} = (\boldsymbol{v}_1 | \boldsymbol{v}_2).$$

It is known that the eigenvector corresponding to distinct eigenvalues are linearly independent, hence $\boldsymbol{P}$ is invertible (can you prove this fact?). Now just note that

$$\boldsymbol{A}\boldsymbol{P} = (\boldsymbol{A}\boldsymbol{v}_1 | \boldsymbol{A}\boldsymbol{v}_2) = (\lambda_1 \boldsymbol{v}_1 | \lambda_2 \boldsymbol{v}_2) = \boldsymbol{P}\boldsymbol{J},$$

which proves the theorem for case (a).

For case (b), assume that there is one real root of characteristic polynomial with the eigenvector $\boldsymbol{v}_1$. Then there is another vector $\boldsymbol{v}_2$, which satisfies

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1,$$

which is linearly independent of $\boldsymbol{v}_1$ (can you prove it?). Now take $\boldsymbol{P} = (\boldsymbol{v}_1 | \boldsymbol{v}_2)$, and

$$\boldsymbol{A}\boldsymbol{P} = (\lambda \boldsymbol{v}_1 | \boldsymbol{v}_1 + \lambda \boldsymbol{v}_2) = \boldsymbol{P}\boldsymbol{J},$$

where $\boldsymbol{J}$ as in (b).

Finally, in case (c) I have $\lambda_{1,2} = \alpha \pm \mathrm{i}\beta$ as eigenvalues and the corresponding eigenvectors $\boldsymbol{v}_1 \pm \mathrm{i}\boldsymbol{v}_2$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are real nonzero vectors. Let me take $\boldsymbol{P} = (\boldsymbol{v}_1 | \boldsymbol{v}_2)$. Since

$$\boldsymbol{A}(\boldsymbol{v}_1 + \mathrm{i}\boldsymbol{v}_2) = (\alpha + \mathrm{i}\beta)(\boldsymbol{v}_1 + \mathrm{i}\boldsymbol{v}_2),$$

I have

$$\boldsymbol{A}\boldsymbol{v}_1 = \alpha\boldsymbol{v}_1 - \beta\boldsymbol{v}_2, \quad \boldsymbol{A}\boldsymbol{v}_2 = \alpha\boldsymbol{v}_2 + \beta\boldsymbol{v}_1.$$

Now

$$\boldsymbol{A}\boldsymbol{P} = (\alpha\boldsymbol{v}_1 - \beta\boldsymbol{v}_2|\beta\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2) = \boldsymbol{P}\boldsymbol{J},$$

where $\boldsymbol{J}$ as in (c). The only missing point is to prove that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, which is left as an exercise. ■

Example 11. Consider system (2) with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}.$$

We find that the eigenvalues and eigenvectors are

$$\lambda_1 = -2, \quad \boldsymbol{v}_1^\top = (-1, 1), \qquad \lambda_2 = 2, \quad \boldsymbol{v}_2^\top = (3, 1).$$

Therefore, the transformation $\boldsymbol{P}$ here is

$$\boldsymbol{P} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix},$$

and

$$\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The solution to system

$$\dot{\boldsymbol{y}} = \boldsymbol{J}\boldsymbol{y},$$

where $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$ is straightforward and given by

$$\boldsymbol{y}(t; \boldsymbol{x}_0) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \boldsymbol{y}_0$$

and its phase portrait has the structure of a saddle (see the figure). To see how actually the phase portrait looks in $\boldsymbol{x}$ coordinate, consider solution for $\boldsymbol{x}$, which takes the form

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} = (\boldsymbol{v}_1 e^{\lambda_1 t}|\boldsymbol{v}_2 e^{\lambda_2 t})\boldsymbol{y}_0 = C_1\boldsymbol{v}_1 e^{\lambda_1 t} + C_2\boldsymbol{v}_2 e^{\lambda_2 t},$$

where I use C_1, C_2 for arbitrary constants. Note that $\boldsymbol{x}$ changing along the straight line with the direction $\boldsymbol{v}_1$ if $C_2 = 0$, and along the straight line $\boldsymbol{v}_2$ when $C_1 = 0$. The directions of the flow on these lines coincide with the directions of the flow on the y-axes for the system with the matrix in the Jordan normal form (see the figure).

This is how we can see the phase portrait for the linear system of two autonomous ODE of the first order. Not taking into account the cases when one or both eigenvalues are zero, we therefore saw all possible phase portraits a linear system (2) can have. Now it is time discuss stability.

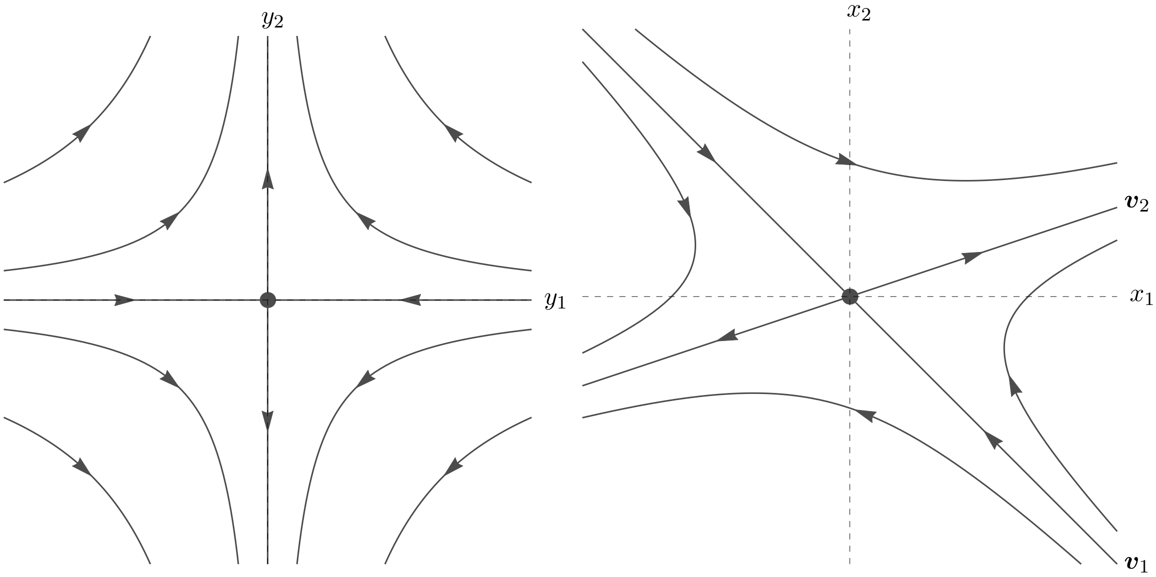

Figure 5: Saddle point after the linear transformation (left), and the original phase portraits (right). The coordinates are connected by the relation $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$, where $\boldsymbol{P}$ is defined in the text

7.4 Stability of the linear system (2)

In what follows I will assume that $\det \boldsymbol{A} \neq 0$, i.e., this means that there is only one isolated equilibrium of system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^2, \tag{2}$$

which is the origin: $\hat{\boldsymbol{x}} = (0, 0)$. To define stability of this equilibrium, and therefore stability of the linear system itself, I need a notion of a neighborhood and distance in the set $\mathbf{R}^2$. I will use the following generalization of the absolute value to vectors $\boldsymbol{x} = (x_1, x_2)^\top \in \mathbf{R}^2$:

$$|\boldsymbol{x}| = \left((x_1)^2 + (x_2)^2\right)^{1/2}.$$

Then distance between two vectors $\boldsymbol{x}_1 \in \mathbf{R}^2$ and $\boldsymbol{x}_2 \in \mathbf{R}^2$ is simply

$$|\boldsymbol{x}_1 - \boldsymbol{x}_2|.$$

Using this convenient notation, now I will verbatim repeat my definition of stability of equilibria of the scalar autonomous ODE. To wit,

Definition 12. *An equilibrium $\hat{\boldsymbol{x}}$ is called Lyapunov stable if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for any initial conditions*

$$|\hat{\boldsymbol{x}} - \boldsymbol{x}_0| < \delta,$$

the flow of (2) *satisfies*

$$|\hat{\boldsymbol{x}} - \boldsymbol{x}(t; \boldsymbol{x}_0)| < \epsilon$$

for any $t > t_0$.

If, additionally,

$$|\hat{\boldsymbol{x}} - \boldsymbol{x}(t;\boldsymbol{x}_0)| \to 0,$$

when $t \to \infty$, then $\hat{\boldsymbol{x}}$ is called asymptotically stable.

If for any initial condition $\boldsymbol{x}_0$ the orbit $\gamma(\boldsymbol{x}_0)$ leaves a neighborhood of $\hat{\boldsymbol{x}}$, then this point is called unstable.

The analysis of linear systems is simple since I actually have the flow of the system given by

$$\boldsymbol{x}(t;\boldsymbol{x}_0) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

Case by case analysis from this lecture allows me to formulate the following theorem:

Theorem 13. *Let $\det \boldsymbol{A} \neq 0$. Then the isolated equilibrium point $\hat{\boldsymbol{x}} = (0,0)$ of planar system (2) is asymptotically stable if and only if for the eigenvalues of $\boldsymbol{A}$ it is true that $\operatorname{Re}\lambda_{1,2} < 0$. If $\operatorname{Re}\lambda_{1,2} = 0$ then the origin is Lyapunov stable, but not asymptotically stable (center). If for at least one eigenvalue it is true that $\operatorname{Re}\lambda_i > 0$ then the origin is unstable.*

Therefore I can have asymptotically stable nodes, improper nodes, and foci, Lyapunov stable center, and unstable nodes, improper nodes, foci, and saddles (note that the latter are always unstable).

I can summarize all the information in one parametric portrait of linear system (2). For this it is useful to consider the characteristic polynomial of $\boldsymbol{A}$ as

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 + \lambda \operatorname{tr}\boldsymbol{A} + \det\boldsymbol{A},$$

where I use the notation $\operatorname{tr}\boldsymbol{A} = a_{11} + a_{22}$ to denote the *trace* of $\boldsymbol{A}$. I have

$$\lambda_{1,2} = \frac{\operatorname{tr}\boldsymbol{A} \pm \sqrt{(\operatorname{tr}\boldsymbol{A})^2 - 4\det\boldsymbol{A}}}{2},$$

therefore the condition for the asymptotic stability becomes simply

$$\operatorname{tr}\boldsymbol{A} < 0, \quad \det\boldsymbol{A} > 0.$$

Using the trace and determinant as new parameters I can actually present possible linear systems as in the following figure

7.5 Bifurcations in the linear systems

I came to the main point of this lecture. I have four parameters in linear systems (2), four entries of matrix $\boldsymbol{A}$. I can consider variations of these parameters, and the structure of the phase portrait of the linear system will be changing. For example, if I cross the boundary $\det\boldsymbol{A} = 0$ in the last figure for negative $\operatorname{tr}\boldsymbol{A}$, then the saddle becomes the stable node. Is this a bifurcation in the system? Or, when crossing the curve $\det\boldsymbol{A} = \frac{1}{4}(\operatorname{tr}\boldsymbol{A})^2$ for positive values of $\operatorname{tr}\boldsymbol{A}$ the unstable focus turns into the unstable node. Is this change enough to call it a bifurcation? Recall that for the first order equations the definition of bifurcation was based on the notion of topological equivalence, which identified equations with the same orbit structure as being topologically equivalent. But now I can have much richer orbit structures because I am not confined any longer to the phase line, now I am dealing with the phase plane.

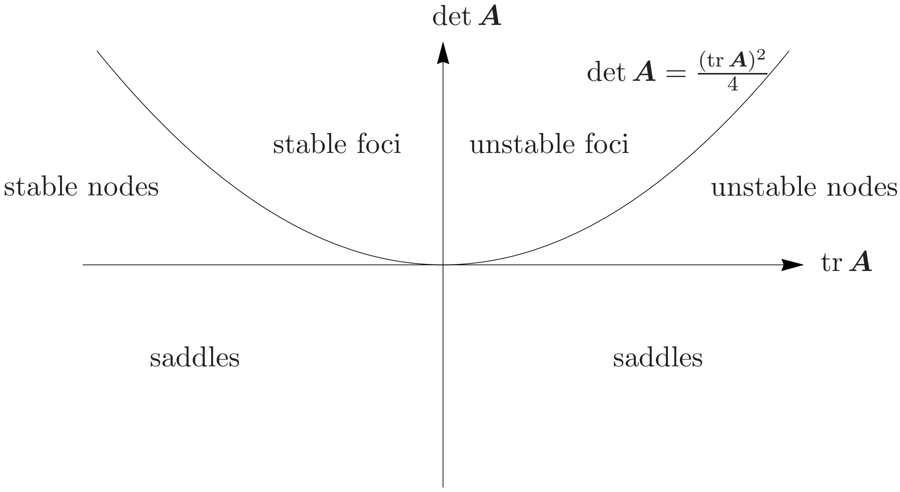

Figure 6: The type of the linear system depending on the values of $\operatorname{tr}\boldsymbol{A}$ and $\det\boldsymbol{A}$. The centers here are situated where $\det\boldsymbol{A} > 0$ and $\operatorname{tr}\boldsymbol{A} = 0$

This is a quite complicated subject, therefore, I will mostly state results, proofs can be found elsewhere.

First, there is a notion of topological equivalence, which includes the one that we already discussed, as a particular case.

Definition 14. *Two planar linear systems $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ and $\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x}$ are called topologically equivalent if there exists a homeomorphism $\boldsymbol{h}\colon \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ of the plane, that is, $\boldsymbol{h}$ is continuous with continuous inverse, that maps the orbits of the first system onto the orbits of the second system preserving the direction of time.*

It can be shown that the notion of topological equivalence is indeed an equivalence relation, i.e., it divides all possible planar linear system into distinct non-intersecting classes.

The following theorem gives the topological classification of the linear planar system. It is also convenient to have

Definition 15. *An equilibrium $\hat{\boldsymbol{x}}$ of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is called hyperbolic if $\operatorname{Re}\lambda_{1,2} \neq 0$, where $\lambda_{1,2}$ are the eigenvalues of $\boldsymbol{A}$. Matrix $\boldsymbol{A}$ as well as system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ itself are also called hyperbolic in this case.*

Theorem 16. *Two linear systems with hyperbolic equilibria are topologically equivalent if and only if the number of eigenvalues with positive real part (and hence the number of eigenvalues with negative real part) is the same for both systems.*

I usually have this large zoo of the equilibria: nodes, saddles, foci, but from topological point of view there are only three non-equivalent classes of hyperbolic equilibria: with two negative eigenvalues (a hyperbolic sink), one negative and one positive (a saddle), and with two positive eigenvalues (a hyperbolic source).

I recommend for a thorough treatment of the subject the following textbook Hale, J. K., & Koçak, H. (2012). Dynamics and bifurcations. Springer

Definition 17. *A bifurcation is a change of the topological type of the system under parameter variation.*

Now I state the result that asserts that it is impossible to have bifurcations in the linear system when the equilibrium is hyperbolic.

Proposition 18. *Let $\boldsymbol{A}$ be hyperbolic. Then there is a neighborhood U of $\boldsymbol{A}$ in $\mathbf{R}^4$ such that for any $\boldsymbol{B} \in U$ system $\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x}$ is topologically equivalent to $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.*

Proof. The eigenvalues as the roots of the characteristic polynomial depend continuously on the entries of $\boldsymbol{A}$. Therefore, for any hyperbolic equilibrium a small enough perturbation of the matrix will lead to the eigenvalues that have the same sign of $\operatorname{Re}\lambda_{1,2}$. Which implies, by Theorem 16, that this new system will be topologically equivalent to the original system. ■

Introducing another term, I can say that a property about 2×2 matrices is *generic* if the set of matrices possessing this property is dense and open in $\mathbf{R}^4$. It can be proved that hyperbolicity is a generic property. I will not go into these details, but rephrase the previous as follows: almost all matrices 2×2 are hyperbolic. This brings an important question: Do we really need to study non-hyperbolic matrices and non-hyperbolic equilibria of linear systems? The point is that, in real applications the parameters of the matrix are the data that we collect in our experiments and observations. These data always have some noise in it, we never know it exactly. Therefore, only hyperbolic systems seem to be observed in the real life. However, this is not the case. Very often, systems under investigation may possess certain symmetries (such as conservation of energy). Another situation, which is more important for our course, that all the system we study contain parameters, whose values we do not know. It is therefore unavoidable that under a continuous parameter change the system matrix will be non-hyperbolic, and at some point we will need to cross the boundary between topologically non equivalent behaviors.

Taking into account the Jordan normal forms for 2×2 matrices, I can consider the following three parameter dependent matrices:

$$(a) \quad \begin{bmatrix} -1 & 0 \\ 0 & \mu \end{bmatrix}, \quad (b) \quad \begin{bmatrix} \mu_1 & 1 \\ \mu_2 & \mu_1 \end{bmatrix}, \quad (c) \quad \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}.$$

These three matrices becomes non-hyperbolic when $\mu = 0$ or $\mu_1 = \mu_2 = 0$. For example in the case (a) when we perturb μ around zero the matrix with $\operatorname{tr}\boldsymbol{A} < 0$ and $\det\boldsymbol{A} < 0$ becomes a matrix with $\operatorname{tr}\boldsymbol{A} < 0$ and $\det\boldsymbol{A} > 0$, i.e., a topological sink turns into a saddle (or vice verse). In the third case I have the change from $\operatorname{tr}\boldsymbol{A} > 0$ and $\det\boldsymbol{A} > 0$ to $\operatorname{tr}\boldsymbol{A} < 0$ and $\det\boldsymbol{A} > 0$, i.e., a topological sink becomes a topological source. In the case (b) the situation is more involved since I have two parameters, and I will not discuss it here. To conclude, I say that for cases (a) and (c) it is enough to have one parameter, and a bifurcation of *codimension* one occurs, whereas for the case (b) I face a *codimension* two bifurcation.